Avalanche dynamics in Bak-Sneppen evolution model observed with standard distribution width of fitness

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Abstract

We introduce the standard distribution width of fitness to characterize the global and individual features of a ecosystem in the Bak-Sneppen evolution model. Through tracking this quantity in evolution, a different hierarchy of avalanche dynamics, w_0 avalanche is observed. The corresponding gap equation and the self-organized threshold w_c are obtained. The critical exponents τ, γ and ρ, which describe the behavior of the avalanche size distribution, the average avalanche size and the relaxation to attractor, respectively, are calculated with numerical simulation. The exact master equation and γ equation are derived. And the scaling relations are established among the critical exponents of this new avalanche.

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I. INTRODUCTION

The biological evolution theory of Darwin is well known to all. The biological evolution can be divided into two different types, micro-evolution and macro-evolution, which focus on the dynamics of evolution at the level of a single population and of all species in ecosystem respectively. Up to now, many scaling laws have been found in macroevolutionary data. The distribution $N(m)$ of extinction sizes for families m decreases with m according to a power-law: $N(m) \propto m^{-\alpha}$ [1]. The distribution $N(t)$ of genera lifetimes t follows $N(t) \propto t^{-k}$ [2]. The distribution of the number of genera $N(g)$ with g species can be fitted with $N(g) \propto g^{-\mu}$ [3]. The disappearance of species or genera is similar to radioactive decay, in which the amount of the "original element" decays exponentially with time. This phenomenon has been named as Red Queen effect [4]. The extinction events exhibit punctuated patterns [5]. The punctuated equilibrium can be interpreted as a metastable state of the dynamics of that single species [6]. If an ecosystem is in equilibrium, every species is stable. However, if one of them fluctuates with time, it will lead to the destabilization of its neighboring species. Thus, the avalanche fluctuation will sweep through the evolution.

The existence of scaling laws in macro-evolution indicates that the dynamics of large-scale evolution is the result of a self-organized critical process [7–9]. Based upon this view, Bak and Sneppen introduced a toy model for species evolution [10]. It is defined and simulated as follows: (i) L species are arranged on a one-dimensional lattice with periodic boundary conditions. (ii) Random numbers f_i, which are chosen from a uniform distribution between 0 and 1, $P(f)$, are independently assigned to each site (species). (iii) At each time step, the extremal site, i.e., the one with the smallest random number $f_{\min}$, and its nearest neighbors are mutated by assigning new random numbers in $P(f)$. As the system evolves it turns out that the species with fitness significantly above the self-organized threshold f_c (equal to 0.66702 [11]) will never mutate, unless its neighbors mutate. After enough mutations have occurred, the ecosystem reaches a statistically stationary state in which the distribution for the fitness of all species is statistically stationary.

Since the Bak-Sneppen model was defined, its punctuated patterns and avalanche dynamics have been investigated extensively. Through observing the fluctuations of the smallest fitness, the model shows punctuated equilibrium behavior, i.e., it self-organizes into a critical state with intermittent coevolutionary avalanches of all size [10]. Paczuski, Maslov and Bak studied the f_0 avalanche (PMB avalanche), forward avalanche and backward avalanche and derived the exact scaling equations, they also established the relations among the critical exponents [11]. Recently, by monitoring the variations of the average fitness, the $\overline{f}_0$ avalanche (LC avalanche), is found, and the exact equations and scaling relations are obtained [12–14]. As we know, the avalanche dynamics is a kind of macroscopic phenomenon in driven dissipative systems. The detailed dynamics of these systems sensitively depend on the initial configurations. But the distribution of avalanches, i.e., scaling law, does not depend on such details due to the universality of complexity [15]. In the existing works, analyzing the smallest fitness only directly connects with the feature of individuals, but it does not directly represent the global feature; discussing the average fitness only directly connects with the global dynamics, but it does not directly manifest the difference in individuals. Can we find a new quantity, which can describe the global feature and the individual difference simultaneously? In this paper, we define such a quantity and observe the corresponding avalanche dynamics with this quantity.

In the next section, the standard distribution width of fitness is defined, and its gap equation is obtained. In Sec. III, the dynamics of avalanche is analyzed in detail, the exact mast equation is derived, and the scaling relations are established among the critical exponents. The last section is a brief summary.

II. STANDARD DISTRIBUTION WIDTH OF FITNESS AND ITS GAP EQUATION

In the Bak-Sneppen model, the fitness of the i-th species is denote as f_i $(i = 1, 2, \cdots, L)$, with L being the total number of species. The fitness represents the population or

the living capability of the species, large fitness means immense population or great living capability, and vice versa. To describe the average population or average living capability of an ecosystem, Li and Cai defined the average fitness as [12–14]

$$f_{av}(s) = \frac{1}{L}\sum_{i=1}^{L} f_i(s). \tag{1}$$

where $f_i(s)$ is the fitness of the i-th species at time s. Based upon the average fitness, we define the standard distribution width of fitness as

$$w(s) = \sqrt{\frac{1}{L}\sum_{i=1}^{L}(f_i(s) - f_{av}(s))^2}. \tag{2}$$

From the above definition, we know that the standard distribution width of fitness simultaneously manifests both global dynamics and individual difference. At the beginning of the evolution, i.e., $s = 0$, the fitnesses are uniformly distributed in $(0, 1)$. Thus for an infinite-size ecosystem ($L \to \infty$), the initial average fitness $f_{av}(0)$ equals 0.5, the initial standard distribution width $w(0)$ is $\sqrt{3}/6$. But for a finite-size ecosystem, $f_{av}(0)$ and $w(0)$ fluctuate around the above values due to the finite size. Because of the absence of mutations or updates, $w(0)$ does not show the correlation between species, only when the evolution goes on, such correlation is contained in $w(s)$ gradually.

As time increases, $f_{av}(s)$ shows an increasing tendency [12], whereas, $w(s)$ shows a decreasing tendency (see the bottom of Fig. 1). After a long transience, i.e., $s \to \infty$, the ecosystem self-organizes to a critical state, and, $f_{av}(s)$ and $w(s)$ approach to different thresholds. When $s \to \infty, L \to \infty$, almost all species have fitnesses above the threshold $f_c = 0.66702$ [11], and the fitnesses are uniformly distributed in $(f_c, 1)$. With the definitions of $f_{av}(s)$ and $w(s)$, we obtain the relation among the thresholds

$$w_c = \frac{1}{2\sqrt{3}}(1 - f_c) = \frac{1}{\sqrt{3}}(1 - f_{avc}). \tag{3}$$

Here, w_c and f_{avc} are thresholds of $w(s)$ and $f_{av}(s)$ respectively. Thus, one gets $w_c = 0.09623$ and $f_{avc} = 0.83351$. For the finite-size ecosystem, the self-organized critical thresholds will fluctuate around the above values, and this fluctuation decreases with the increase of L.

Similar to the definition in Refs. [11,12], we define the gap of $w(s)$ as its envelope:

$$W(s) = \begin{cases} w(0) & , \ s = 0, \\ \min(w(s')|s' \in [0, s]) & , \ s > 0. \end{cases} \tag{4}$$

The above definition means that the current gap of $w(s)$ is the most minimum of all $w(s')$ ($0 \leq s' \leq s$). In the top of Fig. 1, we show the gap $W(s)$ as a function of s. With the increase of time, $W(s)$ gradually approaches close to the threshold. By definition, when the gap falls to its next lower value, the separate instances are separated by avalanches. The avalanches correspond to plateaus in $W(s)$ during which $w(s) > W(s)$. A new avalanche is initiated when the gap falls, and the old avalanche terminates at the same time. Once a avalanche is over it will never affect the behavior of any subsequent avalanche. When the gap decreases, the probability for $w(s)$ to jump above $W(s)$ increases, thus longer and longer avalanches happen typically. Finally, the ecosystem falls into statistically stationary state.

Following Refs. [11,12], we derive the gap equation of $W(s)$. If the gap is changed from $W(s)$ to $W(s)+\Delta W$, the average number of avalanches occurred is $N_{av} = -\Delta W L/(W(s) - w_c)$. Thus, when $L \to \infty$, the average number of time steps required for increasing the gap from $W(s)$ to $W(s) + \Delta W$ is $\Delta s = \langle S \rangle_{W(s)} N_{av} = - \langle S \rangle_{W(s)} \Delta W L/(W(s) - w_c)$, where $\langle S \rangle_{W(s)}$ is the average size of avalanche of the current plateau of the gap. So the differential equation for $W(s)$ is

$$\frac{dW(s)}{ds} = \lim_{\Delta W, \Delta s \to 0} \frac{\Delta W}{\Delta s} = -\frac{W(s) - w_c}{\langle S \rangle_{W(s)} L}. \tag{5}$$

Eq.(5) describes the relaxation to attractor. If the law of $\langle S \rangle_{W(s)}$ is obtained, one can derive the law of relaxation to attractor from the above equation.

III. AVALANCHE DYNAMICS

We know that all self-organized critical systems will exhibit a power-law avalanche dynamics. This has been confirmed by a lot of models with self-organized criticality [10–14,16–22]. In this section, based on the definition of PMB avalanche [11] and LC

avalanche [12], the definition of a new avalanche, w_0 avalanche, is presented. Then its critical exponents are calculated with numerical simulation and its exact master equation is derived. Lastly, the scaling relations are established.

The definition for w_0 avalanche is as follows. Similar to those used in Refs. [11,12], we introduce an auxiliary parameter w_0, where $w(0) > w_0 > w_c$. Suppose that at time step s_1, the current standard distribution width $w(s_1)$ is less than w_0. If the next standard distribution width $w(s_1+1)$ is larger than w_0, a creation-annihilation branching process is initiated. The avalanche will continue to run unless $w(s)$ becomes less than w_0. This means that, at time s, if all $w(s') > w_0$ for $(s_1 < s' < s-1)$, the current avalanche continues. In terms of the above definition, the size of an avalanche is the number of time steps of subsequent punctuations above w_0. If the first appearance of $w(s) < w_0$ after time s_1 occurs at time $s_1 + S$, the size of the current avalanche is S.

Clearly, the above definition ensures the hierarchical structure, i.e., larger avalanches consist of smaller ones. With the decreasing of w_0, smaller avalanches combine into larger ones; if w_0 is set as w_c, the infinite-size avalanche appears. In the other case, with the increasing of w_0, larger avalanches split into smaller ones. Thus, the cutoff effects is unavoidable when w_0 is not chosen as w_c. Nevertheless, the same scaling laws can be obtained when w_0 is very close to the threshold w_c. For the Bak-Sneppen model with 200 species, choosing w_0 as 0.1100, we find by numerical simulation that the distribution of avalanche size follows the power law

$$P(S) \propto S^{-\tau}, \tag{6}$$

where $P(S)$ is the probability of avalanche of size S, τ equals 1.63 ± 0.06 (see Fig. 2). Because of the cutoff effects, the average avalanche size varies when w_0 changes. Similar to those in Refs. [11,13,14], our numerical results show that the average size of w_0 avalanche obeys

$$\langle S \rangle_{w_0} \propto (w_0 - w_c)^{-\gamma}. \tag{7}$$

Here, w_c is the threshold, the exponent γ equals 2.57 ± 0.05 (see Fig. 3). This law means that the divergence of the average avalanche size satisfies power law. Our numerical simulation also shows that the relaxation to attractor abides

$$(W - w_c) \propto s^{-\rho}. \tag{8}$$

Here, s is the time steps, the exponent ρ equals 0.39 ± 0.03. Obviously, the value of ρ is very close to $1/\gamma$.

So far, we have found some scaling laws of the w_0 avalanche by numerically simulations. Now, we will give some exact results. To describe the cascade process of smaller avalanches combining into larger ones when the parameter w_0 is changed, an exact master equation is derived as follows. Denoting the probability of w_0 avalanche of size S as $P(S, w_0)$, when parameter changes from w_0 into $w_0 + \Delta w_0$, some w_0 avalanches merge to larger $w_0 + \Delta w_0$ avalanches, and some small avalanches merge into w_0 avalanche. Thus, at the same time, the probability of a avalanche with a given size flows in and out. Because the termination of w_0 avalanche is uncorrelated, the probability of a w_0 avalanche merging to $w_0 + \Delta w_0$ avalanche is proportional to $-\Delta w_0/(w_0 - w_c)$ [12,13]. Thus the probability which flows out from $P(S, w_0)$ can be written as

$$\Delta_{out} P = -\lambda P(S, w_0) \Delta w_0/(w_0 - w_c), \tag{9}$$

where λ is a constant. The probability which flows in $P(S, w_0)$ is

$$\Delta_{in} P = -\lambda \sum_{S_1=1}^{S-1} P(S_1, w_0) P(S - S_1, w_0) \Delta w_0/(w_0 - w_c). \tag{10}$$

When w_0 approaches w_c and Δw_0 approaches zero, we obtain the following master equation

$$-(w_0 - w_c) \frac{\partial P(S, w_0)}{\partial w_0} = -\lambda P(S, w_0) + \lambda \sum_{S_1=1}^{S-1} P(S_1, w_0) P(S - S_1, w_0). \tag{11}$$

As pointed out in the previous section, with the increasing of time, the standard distribution width shows a decreasing tendency. So, when w_0 decreases, the first term on the right side of the master equation reflects the loss of S-size avalanche, and the second term depicts the gain of S-size avalanche.

Similar to Ref. [13], after defining a new variable $u = -\ln(w_0 - w_c)$, the master equation becomes

$$\frac{\partial P(S,u)}{\partial u} = -\lambda P(S,u) + \lambda \sum_{S_1=1}^{S-1} P(S_1,u)P(S-S_1,u). \tag{12}$$

Through performing a Laplacian transformation for $P(S,u)$, i.e.,

$$p(\alpha,u) = \sum_{S=1}^{\infty} P(S,u)e^{-\alpha S}, \tag{13}$$

thus the master equation reads as

$$\frac{\partial p(\alpha,u)}{\partial u} = \lambda p(\alpha,u)[p(\alpha,u)-1]. \tag{14}$$

When $\alpha = 0$, $p(\alpha,u) = \sum\limits_{S=1}^{\infty} P(S,u)e^{-\alpha S} = 1$, this is the normalization of $P(S,u)$. While $\alpha > 0$, one gets $0 < p(\alpha,u) < 1$. Expanding both sides of Eq.(14) as Taylor series throughout the neighborhood of the point $\alpha = 0$, one can obtain

$$\begin{aligned}
&\frac{\partial}{\partial u}[\alpha \langle S\rangle_u - \frac{1}{2!}\alpha^2 \left\langle S^2\right\rangle_u + \frac{1}{3!}\alpha^3 \left\langle S^3\right\rangle_u \cdots] \\
&= \lambda[1 - \alpha \langle S\rangle_u + \frac{1}{2!}\alpha^2 \left\langle S^2\right\rangle_u - \frac{1}{3!}\alpha^3 \left\langle S^3\right\rangle_u \cdots] \\
&\quad \times[\alpha \langle S\rangle_u - \frac{1}{2!}\alpha^2 \left\langle S^2\right\rangle_u + \frac{1}{3!}\alpha^3 \left\langle S^3\right\rangle_u \cdots],
\end{aligned} \tag{15}$$

where $\left\langle S^k\right\rangle_u = \sum\limits_{S=1}^{\infty} S^k P(S,u)$. Comparing the coefficients of different powers of α in Eq.(15), one gains an infinite series of exact equations. The first exact equations can be expressed as

$$\frac{\partial}{\partial u} \ln \langle S\rangle_u = \lambda. \tag{16}$$

Replacing u as $-\ln(w_0 - w_c)$, we obtain the γ equation

$$\frac{\partial}{\partial w_0} \ln \langle S\rangle_{w_0} = -\frac{\lambda}{w_0 - w_c}. \tag{17}$$

Noting the scaling law $\langle S\rangle_{w_0} \propto (w_0 - w_c)^{-\gamma}$, inserting it into the above equation, one can easily gain

$$\gamma = \lambda. \tag{18}$$

By far, we have obtained three critical exponents of the w_0 avalanche: τ, γ and ρ, which describe the behavior of the avalanche size distribution, the average avalanche size, and the relaxation to attractor, respectively. As listed in Ref. [11], there are three other critical exponents: D, σ and υ. The avalanche dimension D is defined by the following scaling relation

$$S \propto R^D. \tag{19}$$

where S is the avalanche size (temporal duration), R is the spatial extent. The σ and υ are defined by the following scaling laws of cutoff

$$S_{co} \propto (w_0 - w_c)^{-1/\sigma}, \tag{20}$$

$$R_{co} \propto (w_0 - w_c)^{-\upsilon}. \tag{21}$$

The symbols S_{co} and R_{co} are the cutoff of the avalanche size and that of spatial extent of the avalanche.

Although there are six critical exponents, only part of them are independent. After choosing τ and γ as the independent exponents, the other ones can be expressed as functions of them. Inserting the scaling law $\langle S \rangle_{w_0} \propto (w_0 - w_c)^{-\gamma}$ into the gap equation (Eq.(5)), and noting the scaling law $(w_0 - w_c) \propto s^{-\rho}$, one can find

$$\rho = \gamma^{-1}. \tag{22}$$

Integrating the equation $\langle S \rangle_{w_0} = \int_1^{S_{co}} SP(S, w_0)dS$, it is easy to obtain

$$\langle S \rangle_{w_0} = \int_1^{S_{co}} SP(S, w_0)dS = \int_1^{S_{co}} S^{1-\tau}dS \propto S_{co}^{2-\tau} \propto (w_0 - w_c)^{-\frac{2-\tau}{\sigma}}. \tag{23}$$

Comparing with $\langle S \rangle_{w_0} \propto (w_0 - w_c)^{-\gamma}$, one can get

$$\sigma = \frac{2-\tau}{\gamma}. \tag{24}$$

Noting that the average number of the covered sites $\langle n_{cov} \rangle_{w_0}$ covered by a w_0 avalanche scales near the critical point as [11] $\langle n_{cov} \rangle_{w_0} \propto R_{cov} \propto R \propto (w_0 - w_c)^{-1}$and $S \propto R^D$, so $\langle n_{cov} \rangle_{w_0} \propto S^{1/D}$, integrating the equation $\langle n_{cov}(S) \rangle_{w_0} = \int_1^{S_{co}} n_{cov}(S)P(S, w_0)dS$ gives

$$\langle n_{cov}(S)\rangle_{w_0} \propto \int_1^{S_{co}} S^{1/D-\tau} dS \propto (w_0 - w_c)^{-\frac{1-\tau+1/D}{\sigma}}. \tag{25}$$

Thus, the avalanche dimension D can be written as

$$D = \frac{1}{\tau+\sigma-1} = \frac{\gamma}{\gamma(\tau-1)+2-\tau}. \tag{26}$$

Noting $S \propto R^D$ and $R_{co} \propto (w_0 - w_c)^{-\upsilon}$, one can immediately obtain

$$S_{co} \propto R_{co}^D \propto (w_0 - w_c)^{-\upsilon D}. \tag{27}$$

Comparing with $S_{co} \propto (w_0 - w_c)^{-1/\sigma}$, it is easy to know

$$\upsilon = \frac{1}{\sigma D} = 1 + \frac{\gamma(\tau-1)}{2-\tau}. \tag{28}$$

Up to now, the scaling relations among the six critical exponents have been successfully established. The existence of two independent critical exponents means that there are two kinds uncorrelated critical behavior in w_0 avalanche.

IV. SUMMARY

In conclusion, by defining a new quantity, the standard distribution width of fitness, a new avalanche, w_0 avalanche, is observed in the evolution of Bak-Sneppen model. We declare this avalanche as a different hierarchy of avalanche, because firstly, the observed quantity is different from those of Refs. [10,11] and Refs. [12–14]; and secondly, the values of the critical exponents are also different from those found in Refs. [10,11] and Refs. [12–14].

From the definition of the standard distribution width of fitness, we easily obtain the corresponding gap equation and the self-organized threshold. Apparently, with this definition, the global feature and the difference among individuals are simultaneously described by the standard distribution width. So, for the evolutionary distribution with changeless or small changed average values, the standard distribution width is a good quantity to observe the evolutionary dynamics. According to the definition of PMB avalanche and LC avalanche, the definition of the w_0 avalanche is presented. With numerical simulation, the

critical exponents τ, γ and ρ are obtained. Then the exact master equation is derived. From the master equation, the γ equation is immediately obtained. Finally, combining all scaling laws, the scaling relations are successfully established among the critical exponents. With these relations, we find there are only two uncorrelated critical behavior in the w_0 avalanche, for there exist only two independent critical exponents.

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FIGURES

FIG. 1. The standard distribution width and its gap. The above row corresponds to the variation of the gap $W(s)$, the bottom row corresponds to the fluctuation of $w(s)$.

FIG. 2. The power-law of the avalanche size distribution.

FIG. 3. The power-law of the divergence of the average avalanche size.

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